

Spectrum of Discrete Time Periodic signals

* $x(n) = x(n+N)$ is periodic with fundamental period integer N .

* Then Fourier Series is used to represent the spectrum or Frequency representation of $x(n) = x(n+N)$:

$$\begin{aligned}
 & \text{Inverse DTFS or IDTFS} \rightarrow \left\{ \begin{aligned} x(n) &= \sum_{k=0}^{N-1} a_k e^{j \frac{2\pi}{N} kn} \\ a_k &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} kn} \end{aligned} \right. \quad \begin{aligned} 0 \leq n \leq N-1 \\ 0 \leq k \leq N-1 \end{aligned}
 \end{aligned}$$

Discrete-Time Fourier Series (DTFS)

where $x(n+N) = x(n)$ and $a_{k+N} = a_k$

* a_k , for $0 \leq k \leq N-1$ represent spectral line for $x(n)$ in one period. That is, $x(n)$ is periodic and Discrete in time domain and

a_k is Discrete and periodic in Frequency domain.

$$* \quad a_k = |a_k| e^{j \phi_k}$$

Properties

- (1) * if $x(n)$ is real then $\begin{cases} |a_k| = |a_{-k}|, \text{ i.e. even} \\ \angle a_k = -\angle a_{-k}, \text{ i.e. odd} \end{cases}$ functions of frequency.
 * if $x(n)$ is real and even then a_k is real.

Note 1 Index k means frequency $\frac{2\pi k}{N}$ radian per sample

Note 2 Frequency in Hz for any k is

$$\frac{2\pi k}{N T_s} = \frac{2\pi k F_s}{N} \quad \text{where } F_s = \frac{1}{T_s} \text{ is the sampling frequency in Hz, } T_s \text{ is sampling interval in seconds.}$$

Note 3 Index k denotes Harmonic number, whose frequency in radian is

$$\frac{2\pi}{N} k F_s.$$

(2) Linearity

$$\text{If } x_1(n) \xleftrightarrow{\text{DTFS}} a_{1k} \quad \text{and } x_2(n) \xleftrightarrow{\text{DTFS}} a_{2k}$$

Then

$$v(n) = v(n+N) = A x_1(n) + B x_2(n) \xleftrightarrow{\text{DTFS}}$$

$$a_{vk} = A a_{1k} + B a_{2k}$$

Periodic in Frequency with period N .

(3) Time-shift

$$\text{If } x(n) \xleftrightarrow{\text{DTFS}} a_k$$

$$\text{then } v(n) = x(n-m) \xleftrightarrow{\text{DTFS}} \left(e^{-j \frac{2\pi}{N} km} a_k \right) = a_{\omega k}$$

Note 4 All signals here in time and in frequency (Reminder) domains are periodic with fundamental period N . They are all also discrete in Time and in Frequency domains.

(4) Regular Convolution

$$\text{If } x(n) = x(n+N) \xleftrightarrow{\text{DTFS}} a_k$$

$$\text{and } h(n) \xleftrightarrow{\text{DTFT}} H(\Omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n}$$

Not periodic

Then

$$g(n) = x(n) * h(n) \xleftrightarrow{\text{DTFS}} a_{gk} = a_k \cdot H\left(\frac{2\pi}{N} k\right)$$

$$g(n) = g(n+N) \text{ periodic.}$$

(5) Periodic Convolution

$$\text{If } x(n) = x(n+N) \xleftrightarrow{\text{DTFS}} a_{\omega k}$$

$$\text{and } h(n) = h(n+N) \xleftrightarrow{\text{DTFS}} a_{hk}$$

$$\text{then } v(n) = v(n+N) = x(n) \odot h(n) \xleftrightarrow{\text{DTFS}} a_{\omega k} = N a_{\omega k} \cdot a_{hk}$$

(6) Modulation in time (i.e. product of periodic sequences)

If $x(n) \xleftrightarrow{\text{DTFS}} a_{xk}$ and $y(n) \xleftrightarrow{\text{DTFS}} a_{yk}$

Then

$$v(n) = x(n) y(n) \xleftrightarrow{\text{DTFS}} a_{vk} = a_{xk} \textcircled{N} a_{yk}$$

periodic Convolution
in frequency domain.

Ex 1

Find spectrum of the signal

(1) $x(n) = 3 + \cos\left(\frac{\pi}{2}n - \frac{\pi}{3}\right) + e^{-j\left(\frac{\pi}{4}n\right)}$

(2) $x(n) = \{1, 0, -1, 0\}$ that is periodic with period 4.
 $\uparrow$

Sol Part (1):

$$x(n) = 3 + \cos\left(\frac{\pi}{2}n - \frac{\pi}{3}\right) + e^{-j\left(\frac{\pi}{4}n\right)}$$

* Period of constant 3 is $N_1 = 1$

period of $\cos\left(\frac{\pi}{2}n - \frac{\pi}{3}\right)$: $\frac{\pi}{2} = 2\pi f_c \Rightarrow f_c = \frac{1}{4}$

$$\Rightarrow N_c = 4$$

period of $e^{-j\left(\frac{\pi}{4}n\right)}$ is $\frac{\pi}{4} = 2\pi f_e \Rightarrow f_e = \frac{1}{8}$

$$\Rightarrow N_e = 8.$$

Thus period of $x(n)$ is

$$N = l N_c = m N_e \quad \text{where} \quad \frac{N_e}{N_c} = \frac{8}{4} = 2$$

Thus $N = 2 N_c = N_e = 8.$

Now

$$x(n) = 3 + \frac{1}{2} e^{j(\frac{\pi}{2}n - \frac{\pi}{3})} + \frac{1}{2} e^{-j(\frac{\pi}{2}n - \frac{\pi}{3})} + e^{-j(\frac{\pi}{4}n)}$$

Note that we have

$$\begin{aligned} x(n) &= \sum_{k=0}^{N-1} a_k e^{j \frac{2\pi k}{N} n} = \sum_{k=0}^7 a_k e^{j \frac{2\pi k}{8} n} \\ &= a_0 + a_1 e^{j \frac{2\pi}{8} n} + a_2 e^{j \frac{2\pi \cdot 2}{8} n} + a_3 e^{j \frac{2\pi \cdot 3}{8} n} + a_4 e^{j \frac{2\pi \cdot 4}{8} n} \\ &\quad + a_5 e^{j \frac{2\pi \cdot 5}{8} n} + a_6 e^{j \frac{2\pi \cdot 6}{8} n} + a_7 e^{j \frac{2\pi \cdot 7}{8} n} \quad (I) \end{aligned}$$

We write the given $x(n)$ in the problem to look like the above expansion so that a_k are found by comparing the terms with the same exponentials, as follows.

$$x(n) = 3 + \frac{1}{2} e^{-j\frac{\pi}{3}} e^{j\frac{2\pi}{8} 2n} + \frac{1}{2} e^{+j\frac{\pi}{3}} e^{-j(\frac{2\pi}{8} 2n)} + e^{-j(\frac{2\pi}{8} n)}$$

To make exponentials of positive powers:

$$e^{-j(\frac{2\pi}{8} 2n)} = e^{j\frac{2\pi}{8} n(6-8)} = e^{j\frac{2\pi \cdot 6}{8} n} = e^{j2\pi n} = e^{j\frac{2\pi \cdot 6}{8} n}$$

$$\text{and } e^{-j(\frac{2\pi}{8} n)} = e^{j\frac{2\pi}{8} n(7-8)} = e^{j\frac{2\pi \cdot 7}{8} n} = e^{j2\pi n} = e^{j\frac{2\pi \cdot 7}{8} n}$$

$$\text{Thus } x(n) = 3 + \frac{1}{2} e^{-j\frac{\pi}{3}} e^{j\frac{2\pi}{8} 2n} + \frac{1}{2} e^{+j\frac{\pi}{3}} e^{j\frac{2\pi}{8} 6n} + e^{j\frac{2\pi}{8} 7n} \quad (II)$$

Comparing II with I we get:

$$a_0 = 3, \quad a_1 = 0, \quad a_2 = \frac{1}{2} e^{-j\frac{\pi}{3}}, \quad a_3 = 0 = a_4 = a_5$$

$$a_6 = \frac{1}{2} e^{+j\frac{\pi}{3}}, \quad a_7 = 1.$$

Second approach:

We have $x(n) = 3 + \frac{1}{2} e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{+j\frac{\pi}{3}} e^{-j\frac{\pi}{2}n} - j\frac{\pi}{4}n$

$$a_k = \frac{1}{8} \sum_{n=0}^7 x(n) e^{-j\frac{2\pi k}{8}n}$$

We use Linearity property:

(i) $v(n) = 3 \Rightarrow a_{vk} = \frac{1}{8} \sum_{n=0}^7 3 e^{-j\frac{2\pi k}{8}n} = \frac{3}{8} \sum_{n=0}^7 e^{-j\frac{2\pi k}{8}n}$

$$a_{v0} = \frac{3}{8} \frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{\pi}{4}k}} = \frac{3}{8} (8) = 3, \quad a_{vk} = 0 \text{ for other } k.$$

(ii) $g(n) = \frac{1}{2} e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{2}n} \Rightarrow a_{gk} = \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) \sum_{n=0}^7 e^{j\frac{\pi}{2}n} e^{-j\frac{2\pi k}{8}n}$

Thus

$$a_{gk} = \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) \sum_{n=0}^7 e^{+j\left(\frac{\pi}{2} - \frac{\pi}{4}k\right)n} = \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) \frac{1 - e^{j4\pi} e^{-j2\pi k}}{1 - e^{j\frac{\pi}{2}} e^{-j\frac{\pi}{4}k}}$$

$$= \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) \frac{1 - e^{-j2\pi k}}{1 - j e^{-j\frac{\pi}{4}k}} \Big|_{k=2} = \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) \frac{1 - e^{-j4\pi}}{1 - j e^{-j\frac{\pi}{2}}}$$

$$= \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) \frac{0}{0}$$

Then using L'Hopital rule:

$$a_{g2} = \frac{1}{2} e^{-j\frac{\pi}{3}} \left(\frac{1}{8}\right) (8) = \frac{1}{2} e^{-j\frac{\pi}{3}}, \quad a_{gk} = 0 \text{ for other } k.$$

(iii) $h(n) = \frac{1}{2} e^{+j\frac{\pi}{3}} e^{-j\frac{\pi}{2}n} \Rightarrow a_{hk} = \frac{1}{2} e^{+j\frac{\pi}{3}} \left(\frac{1}{8}\right) \sum_{n=0}^7 e^{j\frac{\pi}{2}n} e^{-j\frac{2\pi k}{8}n}$

$$a_{hk} = \frac{1}{2} e^{+j\frac{\pi}{3}} \left(\frac{1}{8}\right) \frac{1 - e^{j2\pi k}}{1 + j e^{-j\frac{\pi}{4}k}} \bigg|_{k=6} = \frac{1}{2} e^{+j\frac{\pi}{3}} \left(\frac{1}{8}\right) - \frac{0}{0}$$

Applying L'Hopital rule:

$$a_{h6} = \frac{1}{2} e^{+j\frac{\pi}{3}} \left(\frac{1}{8}\right) (8) = \frac{1}{2} e^{+j\frac{\pi}{3}}, \quad a_{hk} = 0 \text{ for other } k.$$

$$(iv) \quad p(n) = e^{-j\frac{\pi}{4}n} \Rightarrow a_{pk} = \frac{1}{8} \sum_{n=0}^7 e^{-j\frac{\pi}{4}n} e^{-j\frac{2\pi k}{8}n}$$

$$a_{pk} = \frac{1}{8} \frac{1 - e^{-j\frac{\pi}{4}8} e^{-j2\pi k}}{1 - e^{-j\frac{\pi}{4}} e^{-j\frac{2\pi k}{8}}} = \frac{1}{8} \frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{\pi}{4}} e^{-j\frac{\pi k}{4}}}$$

For $k=7$ we have

$$a_{p7} = \frac{1}{8} \frac{0}{0}$$

Apply L'Hopital rule we get $a_{p7} = 1$ and $a_{pk} = 0$ for other k .

Thus

DTFS for $x(n)$ is

$$a_0 = a_{20} = 3 \quad a_2 = a_{g2} = \frac{1}{2} e^{-j\frac{\pi}{3}}$$

$$a_6 = a_{h6} = \frac{1}{2} e^{+j\frac{\pi}{3}}, \quad a_7 = a_{p7} = 1$$

$$\text{and } a_1 = a_3 = a_4 = a_5 = 0$$

Part (2): $x(n) = \{1, 0, -1, 0\}$ with $N=4$.

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi k n}{N}}$$

$$a_k = \frac{1}{8} \left\{ 1 + 0 - e^{-j \frac{2\pi 2}{8} k} + 0 \right\}$$

$$a_k = \frac{1}{8} \left\{ 1 - e^{-j \frac{\pi}{2} k} \right\} = \frac{1}{8} \left\{ 1 - (-j)^k \right\}$$

$$a_0 = \frac{1}{8} (0) = 0$$

$$a_1 = \frac{1}{8} \{1 + j\} = \frac{1}{4} e^{j \frac{\pi}{4}}$$

$$a_2 = \frac{1}{8} \{1 + 1\} = \frac{1}{4} e^{j0} = \frac{1}{4}$$

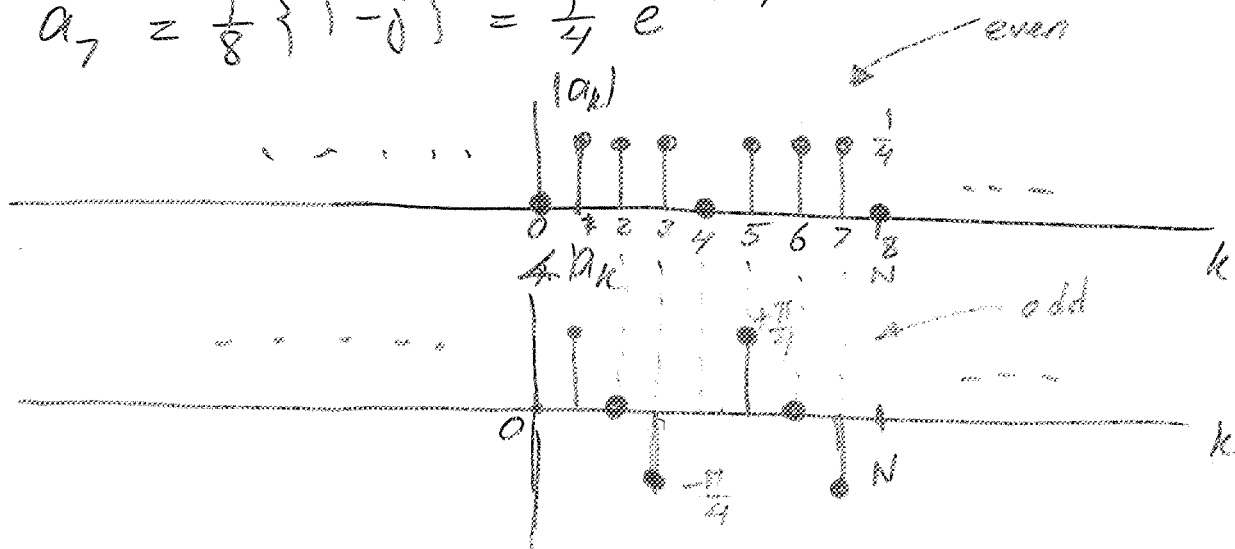
$$a_3 = \frac{1}{8} \{1 - j\} = \frac{1}{4} e^{-j \frac{\pi}{4}}$$

$$a_4 = \frac{1}{8} \{1 - 1\} = 0$$

$$a_5 = \frac{1}{8} \{1 + j\} = \frac{1}{4} e^{j \frac{\pi}{4}}$$

$$a_6 = \frac{1}{8} \{1 + 1\} = \frac{1}{4} = \frac{1}{4} e^{j0}$$

$$a_7 = \frac{1}{8} \{1 - j\} = \frac{1}{4} e^{-j \frac{\pi}{4}}$$



Parseval's Relation

For periodic $x(n) = x(n+N)$ where N is fundamental period:

$$P_{x, \text{av.}} = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |a_k|^2$$

Thus $|a_k|^2$ is Power Density Spectrum or function.

--> See Table in page 221 of Oppenheim book.

Spectrum of Discrete-Time Aperiodic Sequence, $x(n)$

* When $x(n)$ is energy function,

i.e. $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$, i.e. is finite

then spectrum of $x(n)$ is obtained by

Discrete-Time Fourier Transform (DTFT).

$$\begin{aligned} \text{IDTFT} \rightarrow & \begin{cases} x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \\ \text{DTFT} \rightarrow \end{cases} \\ & \begin{cases} X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \end{cases} \end{aligned}$$

$$\begin{cases} \Omega = 2\pi f, & f = \frac{F}{F_s} \leftarrow \text{Frequency in Hz} \\ d\Omega = 2\pi df \end{cases}$$

$T_s = \frac{1}{F_s}$ is sampling interval in seconds. frequency

Thus we can also write

$$\begin{cases} x(n) = \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df, & -\frac{1}{2} < f < \frac{1}{2} \\ X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \end{cases}$$

$$X(\Omega) = |X(\Omega)| e^{j\angle X(\Omega)}$$

Continuous function of Ω , periodic with period 2π .